Problem Sheet 13

Problem 1 (Valuation Rings)

Let A be an integral domain with quotient field K. Show that the following two conditions are equivalent

(a) For every $x \in K^\times$, one has $x \in A$ or $x^{-1} \in A$.

(b) The (a priori only) partially ordered abelian group

$$(\Gamma, \geq) := \big(K^\times/A^\times,\ x \geq y :\Leftrightarrow xy^{-1} \in A\big)$$

is a totally order abelian group.

Such rings are called *valuation rings*. Show that if A is a valuation ring, then

$$v\colon K \to \Gamma \sqcup \{\infty\},\ x \mapsto xA^\times \text{ resp. } 0 \mapsto \infty$$

is a *valuation* meaning that v is "additive" and $v(x+y) \geq \min\{v(x), v(y)\}$.

Problem 2 (Properties of Valuation Rings)

Let A be a valuation ring with quotient field K. Prove that

(a) For any two ideals $\mathfrak{a}, \mathfrak{a}' \subseteq A$ one has $\mathfrak{a} \subseteq \mathfrak{a}'$ or $\mathfrak{a}' \subseteq \mathfrak{a}$. Hence A is local.

(b) Every finitely generated ideal of A is a principal ideal.

(c) Let $A \subseteq B \subseteq K$ be another ring. Show that B is also valuation and of the form $A_{\mathfrak{p}}$ for a prime ideal $\mathfrak{p} \subseteq A$.

Finally, consider the DVR $\mathbb{C}((y))[[x]]$ and its subring

$$A := \left\{ \sum_{n \geq 0} f_n(y) x^n \mid f_0 \in \mathbb{C}[[y]] \right\}.$$

Show that A is valuation and determine the ordered group $K^\times/A^\times$.

Problem 3

Let $v\colon K \to \mathbb{R} \cup \{\infty\}$ be a valued field. Recall that $\mathrm{NP}(f)\colon \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ denotes the Newton polygon of $f \in K[T]$. (By convention $\mathrm{NP}(f)(x) = \infty$ if there are are no $d, e \in \mathbb{N}$ with $x \in [d, e]$ and $a_d, a_e \neq 0$.) Show that

$$\mathrm{NP}(fg) = \mathrm{NP}(f) \star \mathrm{NP}(g)$$

where the convolution on the right hand side is defined as

$$[\mathrm{NP}(f) \star \mathrm{NP}(g)](x) := \min_{i \in \mathbb{R}} \ \mathrm{NP}(f)(x-i) + \mathrm{NP}(g)(i).$$

Hint: You may extend scalars to $\overline{K}$ first and assume f or g to be linear.

Problem 4

Prove that the completion $\mathbb{C}_p$ of the algebraic closure $\overline{\mathbb{Q}_p}$ is algebraically closed.

Hint: Apply Krasner's Lemma.